Provoking the Opposition

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Abstract

Candidates in primary elections face a trade-off between ideological proximity to the party’s base and the probability of winning the general election. I describe conditions under which an incumbent politician provokes the opposition party into nominating more extreme candidates. By moving away from the center, the incumbent can exploit the divisions in the opposition party and improve her reelection prospects, despite hurting her appeal to the median voter in the general election. I identify primaries as an understudied mechanism that drives political polarization in two-party democracies. Furthermore, the analysis fits the observation that reelection-seeking incumbents sometimes move away from the center towards the end of their first term in office. I show that party gatekeeping drives this behavior: with a richer set of candidates, the effect disappears.

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We’re not running to make a statement. We’re not running to pressure the incumbent to the left. We’re running to win.\footnote{https://www.instagram.com/p/BePOZY1lxCZ/}

— Alexandria Ocasio-Cortez

Don’t let extremists give Trump four more years.\footnote{https://www.facebook.com/JohnHickenlooper/videos/254506664889450/}

— John W. Hickenlooper

1 Introduction

The central feature of primary elections is the tug-of-war between a candidate’s electability and her ideological congruence with the party’s base. The former refers to a candidate’s ability to appeal to the median voter, and therefore her chances of winning in the general election. Electability motivations generally push candidates towards moderation in the primaries (Owen and Grofman, 2006). The latter, ideological congruence, refers to primary voters’ desire to nominate a candidate who will faithfully represent their interests if elected. Because primary voters are different from general election voters, candidates who can better appeal to the median voter tend to be further ideologically from the party’s base (Brady, Han, and Pope, 2007; Hall and Snyder Jr, 2015). Primary voters thus face a trade-off between nominating a candidate who has a greater probability of winning in the general election, but less similar ideology; versus someone closer who might have a harder time winning the hearts of swing voters.

I study a forward-looking incumbent politician’s ability to strategically manipulate this trade-off. I propose a model where the incumbent can improve her reelection chances by strategically pursuing extreme policies that provoke and embolden extremist factions in the opposition party. Shifting the incumbent’s position affects the selection of candidates in the opposition, possibly benefiting the incumbent. Thus, I argue that incumbents have both the means and the motivation to influence the opposition party’s primary process.
The core of this paper’s argument is as follows. As the incumbent moves away from the center, the opposition party’s payoffs change in two opposing ways. On the one hand, the value of defeating the incumbent increases. This is because the reelection of a more extreme incumbent would result in worse policies for the members of the opposition, which strengthens their incentive to win the general election and thus pushes them towards the center. On the other hand, a more extreme incumbent increases the probability that any given challenger would win in the general election, including extremists. This emboldens the extremists in the opposition, who see a rare window of opportunity to pursue their agenda. When this latter effect is stronger, a more extreme incumbent leads the opposition party to nominate a candidate ideologically closer to their base, pushing them away from the center. The change in the probability the incumbent is reelected as she becomes more extreme is thus non-monotonic: when the opposition nominates a more extreme challenger, this causes a jump in the probability the incumbent is reelected. I study a model featuring these two opposing forces to discern the conditions under which provoking the opposition occurs and extremism on the one side begets more extremism on the other.

Before presenting this result, I first provide a sufficient condition for the decisiveness of the median primary voter; a result that eluded previous game-theoretic research on primary elections. Proposition 1 proves that when the marginal loss function is log-concave—a condition that is satisfied by virtually every loss function used in the spatial voting literature—the winner of the primary election is the median primary voter’s preferred candidate. This result mirrors the median voter theorem (Hotelling, 1929; Downs, 1957; Black, 1987) for the case of primary elections, and allows us to restrict focus to the ideal point of the median primary voter.

Having established the decisiveness of the median primary voter, I first analyze primary elections under an open nominations model where the opposition is unconstrained in their choice of the ideology of the challenger. I show in Proposition 2 that under open nominations the in-

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3In most papers that study primaries, median primary voter’s decisiveness is assumed either explicitly (Owen and Grofman, 2006; Serra, 2011; Snyder and Ting, 2011) or implicitly (Grofman, Troumpounis, and Xefteris, 2019). Adams and Merrill (2008) assume primary voters ignore the electability of candidates and vote as if they were voting in the general election. Mirhosseini (2015) shows the median primary voter is decisive when losses are quadratic and uncertainty is normally distributed. I generalize this result to much broader classes of loss functions and distributions.
cumbent cannot provoke the opposition by adopting more extreme platforms. This is because an opposition unconstrained in their choice can always respond to a more extreme incumbent by nominating a challenger that improves both their probability of defeating the incumbent and their policy gain.

Of course, primary voters are rarely unconstrained in their choice of politicians. A more realistic model would have primary voters choose from a discrete set of party elites with exogenously given ideologies. I take this approach in the next section and show that—in contrast to the open nominations model—when the opposition has only a discrete set of candidates to pick from, the incumbent can provoke the opposition by choosing platforms more extreme than her ideal point. This happens because a constrained opposition party cannot respond to an incumbent moving to a more extreme platform by producing a candidate that increases both the probability of winning and policy gain. In particular, I show the existence of a threshold platform for the incumbent that induces the median primary voter to switch to supporting a more extreme candidate. By intentionally hurting her appeal to the median voter, the incumbent can induce an extremist challenger to win the opposition party primary and improve her reelection prospects. I show in Proposition 3 the conditions under which the incumbent adopts a platform more extreme than her ideal point to provoke the opposition. This finding highlights that the centripetal force—that politicians move towards the median voter’s ideal point—that is at the heart of much of the spatial voting literature (Wittman, 1983; Calvert, 1985) is reversed when primary elections are taken into consideration.

Next, I consider a candidate-entry model. When deciding whether to run, candidates evaluate the costs of running and the impact their entry has on the eventual winner of the election. Extremists stay out of races against moderate incumbents to let moderate challengers, who are more likely to win, face them in the general election. But they will enter themselves when their chances of beating the incumbent are high enough. Thus, an extremist incumbent induces entry by extremists in the opposition party’s primary, which results in a higher probability of reelection. I show in Proposition 4 that this can occur when the primary field is even or when it is tilted.
in favor of extremists, but not when it is tilted in favor of moderates. Finally, in Proposition 5 I present conditions under which it is preferable for the incumbent to weaken her appeal to the median voter to provoke opposition extremists to enter the race.

2 Related Literature

After the foundational papers studying this trade-off between electability and congruence in the 1970s (Coleman, 1971; Aronson and Ordeshook, 1972), the formal literature on primaries lay mostly dormant until it recently saw a revival (Owen and Grofman, 2006). These theoretical models find that the two opposing forces highlighted in the introduction lead primary voters to support candidates with platforms between their ideal points and that of the population median, conceding slightly on policy for greater electability. When aggregated, this results in candidates who are between the median primary voter and the median general election voter (Jackson, Mathevet, and Mattes, 2007). In other words, the winner of the primary election is an interior solution of this trade-off between electability and ideology; she is not quite so extreme as to be unelectable, and not quite so moderate that she is indistinguishable from the other party’s candidate. This prediction has broad empirical support: according to both observational (Abramowitz, 1989; Abramson et al., 1992; Stone, Rapoport, and Abramowitz, 1992) and experimental studies (Rickershauser and Aldrich, 2007; Woon, 2018), primary voters seem to have both of these considerations in mind while casting ballots, and candidates position themselves accordingly (Brady, Han, and Pope, 2007).

Recent empirical evidence lends support to the idea that there is pressure from both sides. On the one hand, voters with extreme ideologies from either party are both more likely to vote and donate in primary elections than moderates (Hill and Huber, 2017; Barber, Canes-Wrone, and Thrower, 2017). On the other hand, extremists who win primaries are more likely to lose general elections. For example, Hall (2015) finds that when an extremist candidate running for US Congress barely wins a primary, this leads to a smaller vote share and a lower probability
of winning in the general election for their party and that downstream roll-call voting shifts towards the other party. Thus, parties benefit from their opponents electing an extremist; this enables them to retain office and pursue their agenda.

How do parties deal with this trade-off? As others have recognized, we need a better theoretical understanding of this trade-off and the underlying dynamics that drive extremists to win primaries (Tausanovitch and Warshaw, 2018). Many factors play into primary voters’ voting decisions, like the ideological distribution of voters (Serra, 2011; Meirowitz, 2005), the uncertainty about candidates’ abilities to appeal to swing voters (Snyder and Ting, 2011; Adams and Merrill, 2008), concerns about candidates flip-flopping (Hummel, 2010), and the main focus of this paper: the incumbent’s policies (Mirhosseini, 2015). Previous formal literature has identified interesting comparative statics about how the incumbent’s position affects the opposition party’s primaries, but it has done so taking the incumbent’s position as given. Instead, the present paper focuses on a reelection-seeking incumbent’s ability to strategically manipulate the opposition party’s primary voters. It shows how extremists can emerge victorious from primaries endogenously in equilibrium as a result of strategic provocation by the incumbent.

This paper also relates to the literature on candidate entry. Banks and Kiewiet (1989) and Buisseret and Weelden (2020) explore candidate entry to primary races. Thomsen (2014, 2017) finds that more liberal Republicans think they are less likely to win their party’s primary election, and they value winning elections less, relative to more conservative Republicans. This leads to fewer moderate Republicans running for office. Hall (2019) argues that it is the high costs of running, and low benefits for office, that drive moderates out of running. When moderates stay out of politics, extremists—who get more disutility from the other party’s policies—must run themselves to try to prevent the opposition party from winning. The general model of primaries in this paper allows these observations to be recovered. Further, it expands on them by also considering races where extremists join moderates in running for office, despite knowing this increases the probability their party will lose the general election.
3 Primary Elections

I start by presenting the primitives of the model. There is a unidimensional policy space and a measure one of citizens whose ideologies are summarized by their ideal points $x_i \in \mathbb{R}$. Each citizen’s ideology payoff from the implemented policy $x$ is captured by $-\ell(|x - x_i|)$, where the loss function $\ell$ is increasing and convex in the absolute distance between the implemented policy and voter $i$’s ideal point.\(^4\)

I denote by $x_{m_L}, x_{m_R},$ and $x_m$ the ideal points of the median primary voter in $L$, median primary voter in $R$, and the median general election voter respectively. The ideal points of the median primary voters, $x_{m_L}$ and $x_{m_R}$, are common knowledge, with $x_{m_L} \leq 0 \leq x_{m_R}$. Because general electorates are larger and more diverse than primary electorates, and candidates face greater uncertainty regarding the policy preferences and turnout of independent voters, I assume that there is uncertainty about the location of $x_m$. Specifically, $x_m$ is distributed according to some log-concave density $F$ with support that includes $x_{m_L}$ and $x_{m_R}$.

At the start of the game, there is an incumbent affiliated with party $R$ exogenously in power seeking reelection with a platform $x_I \geq 0$. Politicians in the model are also citizens, in that they have ideal points, and receive ideology payoffs given by the loss function $\ell$. The incumbent cares about the policy implemented in the second period only if she is reelected.\(^5\) Both the incumbent and the opposition party candidates enjoy office benefits of $B \geq 0$ if they win the general election. There is no discounting.

The timing of the game is as follows:

1. Incumbent chooses her platform for the second period.

2. The $L$ party primary election is held and a challenger is chosen.

\(^4\)More precisely, I assume $\ell : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is thrice continuously right differentiable, with $\ell(0) = 0$, $\ell' > 0$, and $\ell'' \geq 0$. This class includes all widely used loss functions, including linear, quadratic, and exponential loss.

\(^5\)This simplifies the exposition by stripping the model of motivations for the incumbent to manipulate the ideology of the challenger other than changing her reelection prospects. Results can be recovered if the incumbent’s payoff function includes a continuation value that takes into account her successor’s ideology. Although not without loss of generality, this assumption is standard in the literature. An exception is Bernhardt et al. (2009) who show such concerns push incumbents towards moderation.
3. The general election is held between the incumbent and the challenger.

4. The winner of the general election implements her platform in the second period.

I focus on Subgame Perfect Nash Equilibria in undominated strategies where voters vote for one of the candidates with the lowest platform when they are indifferent between multiple candidates. Ties are resolved by coin flips.

I proceed by backward induction. Analysis of the general election is straightforward. In the second period, the winner of the general election implements her platform. Knowing this, general election voters vote for the candidate who would give them a higher payoff in the second period. Formally, given a challenger $J$ with platform $x_J$, voter $x_i$ votes for the challenger if and only if $-\ell(|x_J - x_i|) \geq -\ell(|x_I - x_i|)$. This means that all voters whose ideal points are to the left of the midpoint between the challenger and the incumbent, $\frac{x_I + x_J}{2}$, vote for the challenger, and those to the right vote for the incumbent. In particular, the median voter votes for the challenger with probability $F \left( \frac{x_I + x_J}{2} \right)$. Because this is an election between two candidates on a single policy dimension with single-peaked preferences, the median voter is decisive. Thus, this expression equals the probability that a challenger with platform $x_J$ defeats an incumbent seeking reelection on platform $x_I$. Having described these probabilities, we are now ready to explore the primary election stage.

3.1 Decisiveness in Primaries

Before analyzing the incumbent’s influence on the choice of challengers in the opposition party, I first present a technical result regarding preference aggregation in primaries. This analogue of the median voter theorem for primary elections is non-trivial, because primary voters face a fundamentally different problem than general election voters. While the latter choose from a set of politicians, primary voters choose from a set of lotteries over politicians. This means that despite single-peaked preferences over a single policy dimension, the decisiveness of the median primary voter does not readily obtain: centrist and extremist primary voters may vote together against the median primary voter. Without a decisive median voter, the previous literature ab-
stracted away from the aggregation problem within parties. Instead, it focused on party insiders as unitary actors unilaterally choosing candidates—which has become less empirically true over the 20th century (Shafer, 2014)—, or by assuming that the median primary voter must be decisive. In this section, I show that the decisiveness of the median primary voter holds whenever the marginal loss function is log-concave, a condition satisfied by virtually all loss functions used in the spatial voting literature.

In principle, there are two ways centrist and extremist primary voters may support a candidate against the one favored by the median primary voter. First, centrist primary voters may join extremists to vote for an extremist challenger. Despite having single-peaked preferences over ideologies, centrist primary voters may not have single-peaked preferences over challenger platforms. Such primary voters may rather vote for an extremist challenger over a moderate one in the primary election to improve the incumbent’s chance of reelection, even though they would prefer the moderate challenger over the extremist to be in office. While the extremist challenger would give them a worse payoff if elected, her probability of being elected is also very low. In contrast, a moderate challenger can both have a high probability of beating the incumbent and result in a much worse payoff than her. Such voters, if they are allowed to vote in the primary, would vote for an extremist candidate to sabotage the party whose primary they are voting in wins in the general election. This is known as “raiding” or “strategic crossover voting” (Chen and Yang, 2002; Oak, 2006).

Second, extremist primary voters may side with centrists to vote for a candidate more moderate than the median primary voter’s preferred candidate. This is because voters’ induced rankings over candidates may be non-monotonic in their ideal points. When a primary voter who is slightly more extreme than another receives disproportionately greater disutility from the incumbent’s reelection, he may be willing to support a more moderate candidate who has a higher chance of beating the incumbent, even if her platform is only slightly closer than that of the incumbent. This can lead to extremist voters joining centrists in supporting a candidate more
moderate than the median primary voter’s optimal candidate.\textsuperscript{6}

A sufficient condition for the existence of a Condorcet winner is that the rate of change of the marginal loss function to not increase too fast. Specifically, I require that the first derivative of the loss function be log-concave: $\ell'' \ell' \leq (\ell'')^2$, where $\ell'$, $\ell''$, and $\ell'''$ respectively refer to the first, second, and third derivatives of the loss function. This assumption, which I maintain going forward, is satisfied by all widely used loss functions including linear, quadratic, and exponential; and precludes the above-mentioned pathological cases by ensuring primary voters’ payoff functions satisfy an “increasing ratios” property in their ideal points and candidates’ platforms.\textsuperscript{7}

The Condorcet winner platform is then the median primary voter’s optimal candidate, $x_{mL}^*$.

**Proposition 1.** The median primary voter’s optimal candidate is the Condorcet winner.

*Proof.* All proofs in Appendix A. \hfill \Box

Proposition 1 shows that under a mild condition satisfied by a broad range of loss functions, the median primary voter is decisive.\textsuperscript{8} A substantive implication of this is that if voters’ policy preferences satisfy the increases ratios property, raiders cannot influence outcomes of primaries.

This is in line with the empirical literature, which finds openness of primaries has little effect on the ideology of the elected legislators (McGhee et al., 2014), or the platforms they run on (Rogowski and Langella, 2015).

Of course, the exact location of the Condorcet winner depends on the functional forms. An interesting case obtains when losses are linear or exponential, $\ell(|x - x_i|) = |x - x_i|$, or $\ell(|x - x_i|) = (x - x_i)^2$ and uncertainty about voter preferences are captured by the normal distribution.

\textsuperscript{6}For example, suppose there are three primary voters whose ideal points are $-6$, $-1$, and $-0.5$ with preferences described by the following loss function

$$
\ell(x - x_i) = \begin{cases} 
|x - x_i| & \text{if } |x - x_i| \leq 5 \\
(x - x_i)^2 - 9(x - x_i) + 25 & \text{otherwise.}
\end{cases}
$$

This function is increasing, convex, and right differentiable. Suppose further that the ideal point of the general election median voter is drawn from a standard normal distribution, and the incumbent’s platform is $x_I = 3$. Here, the median primary voter’s optimal candidate, located at $-1$, loses the primary election against a candidate near $-0.5$, who is preferred by both the centrist and the extremist primary voter.

\textsuperscript{7}The interested reader can compare this property with increasing differences (Ashworth and Bueno de Mesquita, 2006), log-supermodularity (Milgrom and Roberts, 1990), and ratio dominance (Kartik, Lee, and Rappoport, 2019).

\textsuperscript{8}This result is a generalization of Proposition 1 in Mirhosseini (2015) who shows that the median primary voter is decisive when losses are quadratic: $\ell(|x - x_i|) = (x - x_i)^2$ and uncertainty about voter preferences are captured by the normal distribution.
Figure 1: Optimal candidates of $L$ voters when losses are linear or exponential, and $x_{mL} > x^*$. $e^{-|x-x_i|}$. Then, there exists a platform $x^*$ who is the optimal candidate of everyone to her left. Proposition 1 implies that the Condorcet winner must be either $x^*$, or the median primary voter’s ideology, whichever is more moderate:

**Corollary 1.** When losses are linear or exponential, there exists a unique candidate platform $x^*$ such that all voters with ideal points more extreme than $x^*$ prefer a candidate with this platform to face off the incumbent in the general election over any other candidate. Then, the Condorcet winner is the more moderate of $x_{mL}$ and $x^*$.

Thus, for the special cases of linear and exponential losses, there exists a platform $x^*$ such that all primary voters with more extreme ideal points agree that a candidate with that ideology optimally trades off electability for ideology. That is to say, for voters with ideologies $x \leq x^*$, the gain in the probability of winning with someone slightly more moderate than $x^*$ would be too little to justify the ideological loss, and someone slightly more extreme would be too unlikely to win to make the ideology gain worthwhile.\(^9\)

Having established that the median primary voter is decisive under log-concavity of the marginal loss function, I now turn to the case of open nominations, where the opposition party can freely choose a challenger from the ideological spectrum.

### 3.2 Open Nominations

When choosing the ideology of the challenger $x \in \mathbb{R}$, primary voters care not only about the positions of candidates but also their probability of beating the incumbent. Formally, primary

\(^9\)This result is a generalization of Theorem 1 of Owen and Grofman (2006), which shows this for the case of exponential loss and normal $f$.\]
voters solve:

\[
\max_{x \in \mathbb{R}} \text{EU}_i(x, x_I) = \max_{x \in \mathbb{R}} -\ell(|x_i - x|)F\left(\frac{x_I + x}{2}\right) - \ell(x_I - x_i)\left(1 - F\left(\frac{x_I + x}{2}\right)\right).
\]

(1)

The solution to the above problem must be between primary voters’ ideal points and the incumbent’s platform, \(x_i \leq x \leq x_I\). To see why notice first that a candidate with platform \(x_i\) is preferred to anybody more extreme than her. This is because she is both more likely to win the election and results in a higher payoff for primary voters if she does. Furthermore, nominating a candidate whose platform is further right than that of the incumbent is strictly dominated for the primary voters than nominating a candidate to the incumbent’s left.\(^{10}\) These imply that the optimal challenger’s ideology must be in \([x_i, x_I]\). In this interval, we can write the derivative of the primary voters’ problem as:

\[
-\ell'(x - x_i)F\left(\frac{x_I + x}{2}\right) + \frac{1}{2}f\left(\frac{x_I + x}{2}\right)(\ell(x_I - x_i) - \ell(x - x_i))
\]

(2)

The first term in this expression is the marginal loss of policy payoff as the challenger moves away from \(i\)’s ideal point: a more moderate challenger results in a lower payoff for \(i\) if she wins. The second term is the marginal gain from winning with a higher probability. This is positive because a more moderate challenger is more likely to win. Notice that when \(x = x_I\), the second term is zero, which means that this expression is negative. It follows that the optimal challenger platform must be strictly lower than the incumbent’s platform, \(x < x_I\). If the \(\ell'(x - x_i)F\left(\frac{x_I + x}{2}\right)\) term is greater than the second term, this expression is always negative. Substantively, this means that when the probability of defeating the incumbent is high for all \(x\), primary voters always find it worthwhile to trade-off electability for greater ideological congruence. Then the optimal challenger has maximal congruence at \(x_I\). If expression (2) is positive for some platforms and negative for others, there must be a unique interior optimum.

We know from Proposition 1 that the median primary voter’s optimal candidate wins the

\(^{10}\)The arguments sketched here are formally presented and proved in Lemma 3 in the Appendix.
primary. Thus, we can restrict focus to the median primary voter, \( m_L \), and investigate how the optimal challenger ideology changes as a function of the incumbent’s platform. Recall the trade-off faced by primary voters described in the introduction: against a more extreme incumbent, every challenger has a higher chance of winning, because the \( F \left( \frac{x_I + x}{2} \right) \) term increases in \( x_I \). This strengthens the incentives \( m_L \) has to push forward candidates whose platforms he likes more.

On the other hand, a more extreme incumbent increases the value of defeating the incumbent by increasing the \( \ell(x_I - x_{mL}) - \ell(x - x_{mL}) \) component of the payoff function, which pushes him to favor more electable candidates.

Shapes of \( \ell \) and \( F \) determine whether the optimal challenger becomes more or less extreme as the incumbent moves away from the median voter’s ideal point. If a slightly more extreme incumbent causes a much greater policy loss for the median primary voter without causing a big change in the probability of reelection, he would rather choose a more moderate candidate who has a better chance of beating a more extreme incumbent. In contrast, if a slightly more extreme incumbent leads to a substantially lower chance of reelection without causing much additional disutility for the median primary voter, he would rather choose somebody more congruent, even though such a candidate would be less able to appeal to the median general election voter. Depending on the functional forms, the optimal challenger ideology may become more or less extreme as the incumbent moves away from the center. Nevertheless, we can prove that it never becomes so extreme as to improve the incumbent’s probability of reelection. This is because the opposition party always responds to an incumbent moving away from the center by nominating a candidate who will defeat her with a higher probability. Formally, let \( x^* \) and \( x^{**} \) denote the equilibrium challenger platforms the median primary voter chooses against \( x_I \) and \( x'_I \) respectively, and let \( x_I < x'_I \). Then, \( F \left( \frac{x_I + x^{**}}{2} \right) \geq F \left( \frac{x_I + x^*}{2} \right) \).

**Proposition 2.** When the opposition party can choose the ideology of the challenger from the entire policy space, against a more extreme incumbent they choose a challenger who has a higher probability of winning.

The intuition behind this result is as follows; when the opposition party can choose the ideol-
ogy of the challenger, their best response against a more extreme incumbent improves both the probability component \( F \left( \frac{x_I + x}{2} \right) \), and the policy gain component, \( \ell(x_I - x_i) - \ell(x - x_i), \ i \in \{d, m_L\} \).

This means that even if a more extremist incumbent platform leads the optimal challenger platform to also be more extreme, this shift cannot be so large to lead to an overall lower probability of winning in the general election.

Under open nominations, then, the incumbent does not have an incentive to provoke the opposition by pursuing policies more extreme than her ideal point. Next, I study a model where the party cannot freely choose the ideology of the challenger and instead must choose from an exogenously given set of party elites.

### 3.3 Party Elites

The previous section established that when the opposition party can choose the challenger’s ideology from the entire policy space, they respond to a more extreme incumbent by increasing the probability they win in the general election. Therefore, there is no incentive for the incumbent to provoke the opposition. Fielding candidates, however, is rarely an unconstrained optimization problem. When deciding who should run in the general election, a constraint parties face is the set of politicians that they have. There is much evidence showing that politicians who self-select into the profession (Mattozzi and Merlo, 2008; Dal Bó et al., 2017), and who were not screened out by interest groups and party insiders (Cohen et al., 2009; Masket, 2011) are not necessarily representative of the larger population they are drawn from. Furthermore, politicians themselves are constrained in their policy platforms by their previous records. A more realistic model thus would have the opposition party choose from a set of candidates with exogenously given ideologies. This is the approach I take in this section.

Suppose that at the start of the game there is a pair of candidates, Extremist and Moderate, whose platforms are given exogenously. Let their platforms be \( x_E < x_M \leq 0 \). Here, primary voter \( i \)'s problem is to vote for the candidate that gives him a higher expected payoff. An implication of Proposition 1 is that the median primary voter is decisive in the primary between \( E \) and \( M \). The
winner and therefore the challenger against the incumbent is then $E$ if and only if she provides a higher expected payoff to $m_L$ than $M$. This is true whenever $\Delta_{m_L}(x_I) > 0$, where

$$\Delta_{m_L}(x_I) := F \left( \frac{x_I + x_E}{2} \right) \left( \ell(x_I - x_{m_L}) - \ell(|x_{m_L} - x_E|) \right) - F \left( \frac{x_I + x_M}{2} \right) \left( \ell(x_I - x_{m_L}) - \ell(|x_{m_L} - x_M|) \right).$$

In this setting, a more extreme incumbent can be reelected with a higher probability. This is in contrast to Proposition 2 which establishes that this is impossible when the opposition party can freely choose the ideology of the challenger. When primary voters must choose from a discrete set of exogenously given ideologies, there are discontinuities in how they can respond to changes in the incumbent’s platform. In other words, the only tool the median primary voter has here to respond to changes in the incumbent’s platform is picking one candidate with a given platform over another. This allows for configurations such that the median primary voter chooses a challenger who wins with a lower probability against a more extreme incumbent.

Let us first consider the case when the opposition party always nominates the same challenger against all incumbents. If the median primary voter always prefers one candidate over another regardless of the incumbent’s platform, the incumbent’s probability of reelection is monotone decreasing in her platform. It follows that there cannot be an incentive to provoke to opposition by moving away from the center. Thus, I restrict attention to the case where the median primary voter’s choice of $E$ or $M$ is responsive to the incumbent’s position: he prefers $E$ to become the challenger against some incumbents, and $M$ against others. Specifically, I suppose that the median primary voter is ideologically closer to $E$, but $M$’s probability of beating a very moderate incumbent is sufficiently higher:

**Assumption 1. (Responsiveness):** $x_{m_L} < \frac{x_M + x_E}{2}$, and $\Delta_{m_L}(0) < 0$.

Under Assumption 1, incumbents close to the center induce the median primary voter to vote for $M$, and those who are far induce him to vote for $E$. Because $f$ and $\ell$ are both continuous in $x_I \in \mathbb{R}_+$, the median primary voter’s payoff is also continuous. Then, there exists a platform for the incumbent such that the median primary voter is indifferent between $E$ and $M$. Let $x_I$
denote this platform so that when the incumbent’s platform is \( x_I < \tilde{x}_I \), the opposition nominates \( M \) and otherwise nominates \( E \). Allowing for multiple incumbent platforms that leave the median primary voter indifferent complicates the analysis, but does not lead to additional substantively meaningful insights. To simplify exposition, I assume this platform is unique:

**Assumption 2.** *(Single-crossing):* 
\[
\frac{\partial m_I(x_I)}{\partial x_I} \geq 0 \quad \text{for} \quad x_I > \tilde{x}_I.
\]

Assumption 2 is a sufficient condition for single-crossing of the median voter’s net payoff. It states that once the incumbent’s platform is extreme enough for the median primary voter to prefer the extremist, further moves away from the center cannot induce him to switch back to preferring the moderate.

Next, define \( x_I \) as the incumbent platform that leads to the incumbent’s reelection against the moderate challenger \( M \) with the same probability as \( \tilde{x}_I \) wins against \( x_E \). Formally, let 
\[
F\left(\frac{x_I + x_M}{2}\right) = F\left(\frac{\tilde{x}_I + x_E}{2}\right),
\]

if such a platform exists. Otherwise, let \( x_I := 0 \). Incumbents with platforms in the interval \((x_I, \tilde{x}_I)\) face the moderate opponent in the general election and are reelected with a lower probability than incumbents with the more extreme platform \( \tilde{x}_I \) who face the extremist.

**Lemma 1.** *When the median primary voter’s ideal point satisfies Assumptions 1 and 2, there exists an interval of incumbent platforms that result in her facing the moderate opponent and being reelected with a lower probability than if she chose the more extreme platform \( \tilde{x}_I \) and faced the extremist.***

Lemma 1 finds that an incumbent with a more extreme platform can be reelected with a higher probability when the challenger is chosen from a discrete set. To see that this can indeed cause the incumbent to pursue platforms more extreme than her ideal point, observe that her expected payoff is

\[
\max_{x_I \in \mathbb{R}^+} E(U_I(x_I)) = \max_{x_I \in \mathbb{R}^+} \left(1 - F\left(\frac{x_I + x_J}{2}\right)\right) (B - \ell(|x_I - t|)).
\]

where \( J \) is the challenger chosen by \( m_L \) in equilibrium. Solving the incumbent’s problem reveals the conditions under which she provokes the opposition. Specifically, for incumbents whose ideal
Figure 2: Incumbent’s probability of reelection is plotted as a function of her platform. The median voter’s ideal point is standard normally distributed. Losses are linear on the left plot and quadratic on the right. Parameter values are $x_E = -3$, $x_M = -0.5$, and $x_{mL} = -5$.

points lie in the interval $t \in \left( \frac{x_I + \tilde{x}_I}{2}, \tilde{x}_I \right)$, there exist some office rents $B$ such that the incumbent’s optimal platform induces the extremist $E$ to win the opposition party primary, $\tilde{x}_I > t$.

**Proposition 3.** When the incumbent’s ideal point $t$ is more moderate than but sufficiently close to the threshold that induces the extremist opposition candidate to win the primary, there exists an interval of office rents $B$ such that the incumbent provokes the opposition by choosing the platform $\tilde{x}_I > t$ for her reelection bid in equilibrium.

Therefore, when the median primary voter chooses the challenger from a set of party elites whose ideologies are given exogenously, there are parameter values such that the incumbent pursues policies more extreme than her ideal point to improve her reelection chances. She does this solely to weaken her appeal to the median voter. This emboldens the median primary voter in the opposition party, inducing him to vote for the extremist $E$ in the primary election whose platform he prefers to that of the moderate $M$. $E$ gathers the votes of all voters to the left of the median primary voter and beats $M$ to face the incumbent in the general election. However, because $E$’s ideology is further from the general election median voter than $M$, her winning the primary causes a boon to the incumbent’s reelection prospects, surpassing the harm caused by the incumbent’s move away from the center.
4 Endogenous Entry

The previous section supposed that opposition candidates always run in primary races. In this section, I study the strategic considerations of party elites with a model of costly entry. I assume that party elites care both about office rents and what policies are implemented. They take into account their probabilities of winning the primary and the general election, as well as the effect their entry has on the equilibrium outcome. To isolate the effect of the incumbent’s position on the candidates’ considerations, I assume here the winner of the primary is decided by the flip of a (possibly biased) coin.

As before, there are two candidates whose platforms are $x_E < x_M \leq 0$. Candidates announce their running decisions sequentially.\textsuperscript{11} If only one candidate runs, she faces the incumbent in the general election. If both candidates run, $E$ wins the primary with probability $p$. If neither candidate runs, the incumbent is reelected. The only decision candidates make is whether to run and when indifferent they choose to run. The cost of running for office is $c \geq 0$. Throughout I assume that this cost is low enough so that each candidate prefers to run when in equilibrium the other is not running: $c \leq F\left(\frac{x_I + x_J}{2}\right)(B + \ell(x_I - x_J))$ for $J \in \{E, M\}$. The game is otherwise identical to the one described in Section 3.

The timing of the endogenous entry game is as follows:

1. Incumbent chooses her platform for the second period.
2. $E$ and $M$ announce their running decisions according to a predetermined sequence.
3. If only one candidate runs, she becomes the challenger. If both run, $E$ becomes the challenger with probability $p$, and $M$ with probability $1 - p$.
4. If there is a challenger, the general election is held between her and the incumbent.
5. The winner of the general election implements her platform in the second period.

I again proceed by backward induction. The second period and general election play are identical to the previous model, so I skip to candidates’ entry decisions. Because it is simpler, I

\textsuperscript{11}This assumption precludes coordination failures and ensures uniqueness. It can be justified by primaries in the US where announcements are made over the course of several months.
start with the limiting case of no cost of running, \( c = 0 \) before analyzing positive costs.

In deciding whether to enter the primary race when the other candidate is running, each candidate evaluates the benefits of running—choosing the policy and obtaining office rents if they win—and the impact of their entry on both the primary and the general elections. In particular, candidates must consider the effect their entry has on the probability that the incumbent is reelected. If by entering the race a candidate increases the probability that the incumbent is reelected, this may induce them to stay out. This is never the case for the moderate candidate: Because the moderate has a higher probability of beating the incumbent than the extremist, \( M \) can always decrease the probability that the incumbent is retained by entering the primary race. It follows that when the cost of running is zero, \( M \) always enters. Formally, this is because when \( E \) is running, \( M \)'s net expected payoff of entering the race versus staying out, \( (1 - p) \Delta M(x_I) \) is positive, where:

\[
\Delta M(x_I) := F\left(\frac{x_I + x_M}{2}\right) (B + \ell(x_I - x_M)) - F\left(\frac{x_I + x_E}{2}\right) (\ell(x_I - x_M) - \ell(x_M - x_E)).
\]

The first term in this expression is the direct effect: by entering, \( M \) increases the probability he wins. The second term is the indirect effect \( M \)'s entry has on the decreased probability \( E \) faces the incumbent in the general election. Because the probability \( M \) beats the incumbent is higher, \( \Delta M(x_I) \) must be positive.

The same does not hold for \( E \); when the moderate is running, extremist’s entry increases the probability the incumbent is reelected. Thus, \( E \) may prefer to stay out of the race if \( M \) has a sufficiently better chance of beating the incumbent in the general election. In other words, similar to primary voters who vote for the moderate candidate despite liking the policies of the extremist more, \( E \) can concede on her policy goals and office rent to help her party win the general election by letting the more electable \( M \) face the incumbent. Formally, when \( M \) is running, the net expected utility of \( E \) of running versus staying out is \( p \Delta E(x_I) \), where:

\[
\Delta E(x_I) := F\left(\frac{x_I + x_E}{2}\right) (B + \ell(x_I - x_E)) - F\left(\frac{x_I + x_M}{2}\right) (\ell(x_I - x_E) - \ell(x_M - x_E)).
\]
As argued above, this expression can be positive or negative. Notice that it must be positive for sufficiently high values of $x_I$: As $F\left(\frac{x_I+x_M}{2}\right) - F\left(\frac{x_I+x_E}{2}\right)$ goes to zero, entering becomes strictly preferred for $E$.

Next, I make the following modification to the second part of Assumption 1 for the case of candidate entry. Assumption 3 ensures that $M$’s probability of beating a moderate incumbent is sufficiently higher than $E$ for $E$ to prefer to stay out when the incumbent is very moderate. Then, there exists an incumbent platform $\tilde{x}_I'$ that leaves the extremist indifferent between entering and staying out:

**Assumption 3.** (Responsiveness): $\Delta_E(0) < 0$.

Similarly, analogous to Assumption 2, Assumption 4 guarantees this platform $\tilde{x}_I'$ is unique:

**Assumption 4.** (Single-crossing): $\frac{d\Delta_E(x_I)}{dx_I} \geq 0$ for $x_I > \tilde{x}_I'$.

Let $x_I'(p)$ be the platform that leads to the same probability of incumbent’s reelection as $\tilde{x}_I'$, if such a point exists. If such a point does not exist, let $x_I'(p) = 0$. We can then prove an analogue of Lemma 1 in Section 3: despite being more moderate, incumbents with platforms in the interval $(x_I'(p), \tilde{x}_I')$ are reelected with a lower probability than incumbents with the platform $\tilde{x}_I'$.

**Lemma 2.** When Assumptions 3 and 4 hold, there exists an interval of incumbent platforms that result in the incumbent facing the moderate opponent and winning reelection with a lower probability than if she had the more extreme platform $\tilde{x}_I'$ and faced the extremist opponent with probability $p$. Because $M$ plays a dominant strategy in equilibrium, this is true regardless of the sequence of announcements.

Lemma 2 extends to positive costs of running for intermediate values of $p$. When $c > 0$ is small and $p$ close to $1/2$, there exists an incumbent platform that induces the extremist opponent to enter the race, hence resulting in a higher probability of reelection than slightly more moderate platforms. Thus, assuming that candidates face a small but positive cost to enter the race does not give us substantively different insights for intermediate values of $p$. 

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Figure 3: Incumbent’s probability of reelection is plotted as a function of her platform. The median voter’s ideal point is standard normally distributed. Losses are linear on the left plot and quadratic on the right. Parameter values are $x_E = -3$, $x_M = -0.5$, $p = 0.5$, $B = 5$, and $c = 1$.

Next, consider a primary field that is slanted in favor of moderates. Suppose $p$ is close to zero, meaning that the moderate is very likely to win a competitive primary. In this case too, the moderate always runs, irrespective of the sequence of candidates’ announcements. The extremist, in contrast, prefers to stay out of the race, even when the incumbent’s platform is extreme. This is because even if $E$ knew she would likely beat the incumbent in the general election, it is unlikely she can get there, so she decides to stay out of the race. It follows that when the primary field is slanted towards moderates, $M$ can drive $E$ out of running, and the general election is held between the moderate challenger and the incumbent, regardless of the latter’s platform.

Finally, suppose there is an extremist advantage in the primary, or $p$ close to one. Here, the moderate only enters the race if the extremist does not. This is because the probability $M$ makes it through to the general election from a competitive primary field is very low, despite having a higher chance of beating the incumbent if he did. Thus, it is possible for the extremist to be the only candidate in equilibrium. If the incumbent is sufficiently extreme, a strong primary advantage induces the extremist to enter the race even if the moderate were running. That in equilibrium the extremist will enter the race regardless drives the moderate out. In contrast, when $M$ has a much better chance of beating the incumbent, $E$ prefers to stay out and let $M$ face the incumbent in the general election. For intermediate incumbent platforms, who runs in equilibrium depends on the order in which candidates announce their entry decisions. If $M$ announces
Figure 4: Orange (SW) and blue (NE) regions respectively correspond to parameter values where only $M$ and $E$ run in equilibrium. In green (E) both candidates run. The median voter’s ideal point is standard normally distributed. Losses are linear on the left plot and quadratic on the right. Parameter values are $x_E = -1.5$, $x_M = -0.5$, $c = 1.2$, and $B = 4$. In both plots, $M$ announces first. Plots for equilibria when $E$ announces first are presented in Appendix C.

first, he runs if and only if $E$ prefers to stay out when he is running, $\Delta_E(x_I) < \frac{c}{p}$. If $E$ announces first instead, she runs whenever she prefers to face the incumbent herself, $\Delta_E(x_I) \geq c$, knowing that $M$ stays out if she enters. It follows that when there is a primary advantage for extremists, for either sequence of announcements, there exist thresholds such that only the moderate challenger runs against incumbents whose platforms are more moderate than this threshold, and only the extremist challenger runs against those more extreme. These are visualized in Figure 4 and summarized in the following Proposition:

**Proposition 4.** Suppose the cost of running is small but strictly positive. Then,

1. When neither the extremist nor the moderate has an advantage in the primary ($p$ close to $1/2$), there exists a platform for the incumbent that leads to her facing $E$ in the general election with probability $p$ and winning reelection with a higher probability than if she had a more moderate platform and faced $M$ with probability one;

2. When there is a primary advantage for moderates ($p$ close to 0), the incumbent faces $M$ in the general election regardless of her platform;

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3. When there is a primary advantage for extremists (\(p \text{ close to } 1\)), for either sequence of announcements, there exists a platform for the incumbent that leads to her facing \(E\) in the general election and winning reelection with a higher probability than if she had a more moderate platform and faced \(M\).

Proposition 4 finds that when the primary election is slanted in favor of extremists, there cannot be a competitive primary. Against a sufficiently extreme incumbent, \(E\) would rather be the challenger herself to implement her ideal point and obtain office rents, although her probability of winning is lower than the probability \(M\) would win in the general election. This keeps \(M\), who knows he has a very low probability of making it through to the general election, out of the race. In contrast, against a sufficiently moderate incumbent, \(E\) recognizes that she is unlikely to win the general election. In this case, she stays out of the race and the equilibrium challenger is \(M\).

When the primary election is balanced and both sides have roughly equal chances of winning a competitive primary, \(M\) always runs, regardless of the incumbent’s platform and whether \(E\) is also running or not. This is in part because his entry always decreases the probability the incumbent is reelected, and in part because his probability of winning both the primary and the general elections is high enough to cover the cost of entry. In contrast, against a sufficiently moderate incumbent, \(E\) stays out. This is driven by the fact that \(E\) gets a high disutility from the incumbent’s reelection and prefers \(M\) to face and defeat the incumbent with a higher probability. When instead the incumbent’s platform is sufficiently extreme and \(E\) can plausibly beat her in the general election, there is a competitive primary. Here, with probability of about one half, the incumbent faces \(M\), and otherwise faces \(E\) in the general election. Thus, when there is a primary advantage for extremists or no advantage for either side, a more extreme incumbent induces \(E\)’s entry, which increases the overall probability the incumbent is reelected.\(^{12}\)

It follows that an incumbent with an ideal point more moderate than the platform that induces the extremist’s entry may thus find it preferable to pursue this platform. Despite hurting

\(^{12}\)Evidence presented in Hall and Snyder Jr (2015) suggests that these two cases are more relevant than a primary advantage for moderates: They find that extremist candidates tend to have an advantage in primary elections as measured by vote share and probability of winning.
Figure 5: Incumbent’s probability of reelection is plotted as a function of her platform. The median voter’s ideal point is standard normally distributed. Losses are linear on the left plot and quadratic on the right. Parameter values are $x_E = -3$, $x_M = -0.5$, $p = 0.5$, $B = 5$, and $c = 1$. There is a primary advantage for moderates in the top plots ($p = 0.05$), and for extremists in the bottom ($p = 0.95$).

her policy-wise and electorally against any given opponent, going more extreme increases the probability she faces a weaker challenger in the general election. By choosing the threshold platform, the incumbent can increase the probability she faces $E$ in the general election from zero to one if there is a primary advantage for extremists, and to $p$ if there is no primary advantage for either extremists or moderates. For an incumbent with an ideal point sufficiently close to this platform, this leads to a strictly higher expected payoff for appropriate levels of office rents.

**Proposition 5.** Given the ideal points of the moderate and the extremist, $x_M$ and $x_E$; and the cost of running $c$, there exist intervals of office rents $B$ and incumbent ideal points $t$ such that when there is a primary advantage for extremists or when there is no primary advantage for either extremists or moderates, the incumbent provokes the opposition by choosing platforms more extreme than her ideal point. When there is a primary advantage for moderates, provoking the opposition cannot occur.
Proposition 5 shows that, like in the model of primary elections, provoking the opposition is possible in a model of candidate entry. Specifically, moderate incumbents can benefit from hurting themselves electorally by decreasing their appeal to the general election median voter. This increases the probability the extremist opposition party candidate wins in the general election, and hence induces her to run in her party’s primary. If she emerges as the challenger to face the incumbent in the general election, the incumbent is reelected with a higher probability. For incumbents with ideal points close to the threshold that induces extremist’s entry in particular, provoking the opposition leads to a large enough boost to make up for the decreased appeal caused by going more extreme.

5 Conclusion

In this paper, I propose a model of sequential elections where an incumbent politician may find it optimal to pursue extreme policies to improve her reelection prospects. Such policies induce extremist candidates in the opposition party to run for office, and primary voters to support them. I present conditions under which moves to the extreme by the incumbent results in her facing a weaker challenger in the general election, thus improving her chance of reelection. I call this result provoking the opposition, and show it can occur via two distinct mechanisms. First, the incumbent can induce the opposition primary voters to support an extremist candidate whose policies they like more than a more electable centrist when both candidates are running. Second, I show that the incumbent can provoke extremists to run for office, possibly driving moderates out. A natural next question is how these two forces interact. In Appendix B I present simulations that combine these two mechanisms. These suggest that the results are robust to when both primary elections and candidates’ entry decisions are endogenous, and that the effects on primary voters and candidates complement each other. In such a model, more extreme incumbents push voters towards supporting extremist challengers in the primary. This, in turn, makes extremists more likely to run for office and moderates to stay out. Thus, the effects of the incumbent’s move away
from the center on primary voters and challengers reinforce each other, leading to a stronger overall effect on the incumbent’s chances of reelection. I leave a thorough analysis of an explicit model of primary elections with endogenous candidate entry to future research.

An important feature of the model is the incumbent’s ability to commit to a second term platform. She cannot deceive the opposition into thinking she is an extremist, only to back-pedal into a more moderate platform after the opposition’s primary. If this was possible, the incumbent could reap the benefits of provoking the opposition—facing a weak opponent—without suffering an electoral penalty for extremism in the general election. But primary voters would correctly anticipate such a flip-flop and ignore the incumbent’s temporary move away from the center. Thus, the incumbent must be able to commit to a platform before the opposition party primary for the mechanism identified in this paper to work. In contrast to the incumbent, I assume that opposition candidates cannot credibly commit to a platform. Whereas incumbents can signal commitment to platforms other than their ideal point by, for example, passing legislation at odds with their ideologies or nominating incongruent members to the cabinet; the only signals challengers have are cheap talk campaign promises.

The model I present in this paper is based on the idea that an incumbent with a more extreme platform makes it both more likely and more important to defeat her. The first follows from the fact that an extreme platform is further from the ideal point of the median voter, which increases the probability that any given challenger can beat the incumbent. This emboldens the extremist factions within the opposition party who see a window of opportunity to pursue their agenda. On the other hand, an incumbent with an extreme platform also increases the payoff gain of defeating her, because the policy that would be implemented if the incumbent were reelected is disliked more by the members of the opposition. This pushes the opposition towards moderation to increase their appeal to the median voter, and therefore their probability of beating the incumbent. Whether the incumbent can successfully provoke the opposition and improve her reelection chances depends on how these two forces play out, and the set of candidates parties get to choose from. I show that when the opposition party can freely choose the ideology of their
candidate, they always improve the probability of winning against a more extreme incumbent. When parties are constrained in their choice, however, it is possible for an extremist incumbent to win reelection with a higher probability than a moderate one. This highlights a novel implication of parties’ gatekeeping of candidates: the inability of the opposition party to choose challengers from a rich set of candidates enables incumbents to provoke the opposition.

Primaries have been identified as a factor that can exacerbate political polarization by giving ideologically extreme partisans greater say in the nomination process. In this paper, I demonstrate how primaries can also drive moderates in office to adopt extreme positions, leading to even more polarization. This happens as strategic incumbents move away from the center to hurt their appeal to the median voter, and still yet improve their reelection prospects by provoking the opposition to select extremists. Thus, primaries can contribute to the observed proliferation of ideological extremists in contemporary politics in three ways: encouraging extremists to run for office, primary voters to support them, and elected moderates to become more extreme.
References


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Appendix A  Proofs

Proof of Proposition 1. I begin by proving that the optimal challenger platform for a primary voter must be between the voter’s ideal point and the incumbent’s platform:

Lemma 3. For primary voter \( i \), the optimal challenger has an ideal point between his own and the incumbent’s platform: \( x_i \leq x^* \leq x_I \).

Proof of Lemma 3. Start by observing that the challenger cannot be someone whose platform primary voter \( i \) likes less than that of the incumbent, because a candidate whose platform is the same as \( i \)'s ideal point always yields a higher expected payoff than such a candidate. Restricting attention to platforms \( i \) prefers to that of the incumbent: \( |x_i - x_I| > |x_i - x| \), from Equation (1)

\[
EU_i(x, x_i) = -\ell(|x_i - x|)F\left(\frac{x_i + x}{2}\right) - \ell(x_i - x)\left(1 - F\left(\frac{x_i + x}{2}\right)\right).
\]

The derivative of this is

\[
\frac{d}{dx}EU_i = \text{sgn}(x_i - x)\ell'(|x_i - x|)F\left(\frac{x_i + x}{2}\right) + \frac{1}{2}f\left(\frac{x_i + x}{2}\right)\left(\ell(x_i - x) - \ell(|x_i - x|)\right)
\]

where \( \text{sgn}(x - y) := \frac{|x-y|}{x-y} \) for all \( x \neq y \) and zero otherwise. Notice that the second term in this expression is always positive in this region. The first term is also positive, and thus \( i \)'s payoff increasing in \( x \) for \( x < x_i \) because \( \frac{dEU_i}{dx} > 0 \). This implies that \( EU_i(x_i, x_i) > EU_i(x, x_i) \) for all \( x < x_i \). Thus, as argued in the text we can restrict attention to \( x \in [x_i, x_I] \).

Next, notice that the first order condition of the primary voter’s problem is:

\[
F\left(\frac{x_i + x^*}{2}\right) = \frac{\ell(x_i - x_i) - \ell(x^* - x_i)}{2\ell'(x^* - x_i)}.
\]

Both sides of Equation (4) are positive for \( x \in [x_i, x_I] \). Log-concavity of \( f \) implies that the left-hand side is increasing in \( x \). The right-hand side is decreasing in \( x \) because \( \ell \) is increasing and
convex. Thus, there can be at most one solution to Equation (4) in \( x_i \leq x < x_I \). Notice also

\[
d^2 \text{EU}_i \left( \frac{x_i + x}{2} \right) \left( \ell \left( x_I - x_i \right) - \ell \left( x - x_i \right) \right) - f \left( \frac{x_i + x}{2} \right) \ell' \left( x - x_i \right) - F \left( \frac{x_i + x}{2} \right) \ell'' \left( x - x_i \right).
\]

Evaluating this expression at the solution of Equation (4) yields:

\[
\frac{1}{2} f' \left( \frac{x_I + x^*}{2} \right) - \frac{f \left( \frac{x_I + x^*}{2} \right)}{F \left( \frac{x_I + x^*}{2} \right)} - \frac{\ell'' \left( x^* - x_i \right)}{\ell' \left( x^* - x_i \right)} < 0
\]

because log-concavity of \( f \) implies for all \( x \) we have \( f' (x) F(x) < f(x)^2 \). Therefore, if \( x \) solves Equation (4), it is a maximum.

Thus, the median primary voter’s optimal candidate must have a platform between his ideal point and the incumbent’s platform: \( x^*_{m_l} \in [x_{m_l}, x_I] \). To prove that the median primary voter is pivotal, we need to show that every primary voter to his left (right) prefers a candidate with ideology \( x^*_{m_l} \) to any candidate whose ideology is to her right (left). Formally, a sufficient condition for median primary voter’s pivotality is that we have \( \text{EU}_i(x^*_{m_l}, x_I) \geq \text{EU}_i(x, x_I) \) for both all \( x_i \leq x_{m_l} \) and \( x \geq x^*_{m_l} \), and all \( x_i \geq x_{m_l} \) and \( x \leq x^*_{m_l} \). Notice that

\[
\text{EU}_i(x^*_{m_l}, x_I) \geq \text{EU}_i(x, x_I) \iff \frac{\ell(x_I - x_i) - \ell(\left| x^*_{m_l} - x_i \right|)}{\ell(x_I - x_i) - \ell(\left| x - x_i \right|)} \geq \frac{F \left( \frac{x_I + x^*_{m_l}}{2} \right)}{F \left( \frac{x_I + x_{m_l}}{2} \right)}.
\]

By definition of \( x^*_{m_l} \), we know that for all \( x \):

\[
\frac{\ell(x_I - x_i) - \ell(x^*_{m_l} - x_i)}{\ell(x_I - x_i) - \ell(\left| x - x_{m_l} \right|)} \geq \frac{F \left( \frac{x_I + x}{2} \right)}{F \left( \frac{x_I + x^*_{m_l}}{2} \right)}.
\]

So a sufficient condition for the existence of a Condorcet winner is that for \( (x_i - x_{m_l})(x - x^*_{m_l}) \leq 0 \) we have

\[
\frac{\ell(x_I - x_i) - \ell(\left| x^*_{m_l} - x_i \right|)}{\ell(x_I - x_i) - \ell(\left| x - x_i \right|)} \geq \frac{\ell(x_I - x_{m_l}) - \ell(x^*_{m_l} - x_i)}{\ell(x_I - x_{m_l}) - \ell(\left| x - x_{m_l} \right|)}.
\]

To show that the log-concavity of \( \ell' \) is a sufficient condition for the above equality to hold, we
first need to prove the following lemma:

**Lemma 4.** Suppose that $\ell'$ is log-concave. Then, for any $x_i < x_0 < x_1 < x_2$:

\[
\frac{d}{dx_i} \left( \frac{\ell(x_2-x_i) - \ell(x_0-x_i)}{\ell(x_2-x_i) - \ell(x_1-x_i)} \right) \leq 0.
\]

**Proof of Lemma 4.** Suppose that $\ell'$ is log-concave. By definition of log-concavity, this means that for all $x$, we have $\ell'(x)\ell''(x) \leq (\ell''(x))^2$, where $\ell'$, $\ell''$, and $\ell'''$ refer to first, second, and third derivatives of $\ell$ respectively. This in turn implies that the cross-partial of logarithm of $\ell'$ with respect to any $x$ and $x_i$ is positive because

\[
\frac{\partial^2 \ln \ell'(x-x_i)}{\partial x \partial x_i} = \frac{\partial \operatorname{sgn}(x-x_i) \ell''(x-x_i)}{\partial x_i} = -\ell'''(x-x_i)\ell'(x-x_i) + (\ell''(x-x_i))^2 \geq 0,
\]

where the last inequality follows from the log-concavity of $\ell'$. This implies that for any $x_1 > x_0$:

\[
\frac{\partial \ln \ell'(x_1-x_i)}{\partial x_i} - \frac{\partial \ln \ell'(x_0-x_i)}{\partial x_i} \geq 0 \iff \frac{\partial \ln \left( \frac{\ell'(x_1-x_i)}{\ell'(x_0-x_i)} \right)}{\partial x_i} \geq 0 \iff \frac{\partial^2 \ln \ell'(x_1-x_i)}{\partial x_i^2} \geq 0.
\]

Now let $x_j < x_i$, and define $s(x) = \ell'(x-x_i)$ for $x > x_i$. The previous condition implies that $\frac{\partial s(x)}{\partial x} \leq 0$. Let $x_2 > x_1 > x_0$, and notice we can write

\[
\frac{\ell(x_2-x_i) - \ell(x_1-x_i)}{\ell(x_1-x_i) - \ell(x_0-x_i)} = \frac{\int_{x_1}^{x_2} \ell'(x-x_i)dx}{\int_{x_0}^{x_1} \ell'(x-x_i)dx} \geq \frac{\int_{x_1}^{x_2} s(x) \ell'(x-x_i)dx}{\int_{x_0}^{x_1} s(x) \ell'(x-x_i)dx} = \frac{\int_{x_1}^{x_2} \ell'(x-x_i)dx}{\int_{x_0}^{x_1} \ell'(x-x_i)dx} = \frac{\ell(x_2-x_j) - \ell(x_1-x_j)}{\ell(x_1-x_j) - \ell(x_0-x_j)},
\]

where the inequality follows from the fact that $s(x)$ is lower everywhere it’s evaluated in the numerator than everywhere in the denominator. Thus

\[
\frac{\partial^2 \ln \left( \frac{\ell(x_2-x_i)}{\ell(x_1-x_i)} \right)}{\partial x_i^2} \geq 0 \iff \frac{\partial \ln \left( \frac{\ell'(x_1-x_i)}{\ell'(x_0-x_i)} \right)}{\partial x_i} \geq 0 \iff \frac{\partial^2 \ln \ell'(x_1-x_i)}{\partial x_i^2} \geq 0.
\]
Finally, add 1 to the above expression. Because this is a constant, the derivative does not change, and we get
\[
\frac{\partial \ell(x_i - x_i)}{\partial x_i} = \frac{\partial \ell(x_i - x_i - (x_0 - x_1))}{\partial x_i} + 1 = \frac{\partial \ell(x_i - x_i - (x_0 - x_1))}{\partial x_i} \leq 0.
\]
□

We can now use Lemma 4 to show that for \(x \geq x_{mL}^*\), we have
\[
\frac{d \ell(x_i - x_i - (x_0 - x_1))}{dx_i} \leq 0.
\]
Similarly, for \(x \leq x_{mL}^*\) we have
\[
\frac{d \ell(x_i - x_i - (x_0 - x_1))}{dx_i} \geq 0.
\]

It follows that
\[
\frac{\ell(x_i - x_i) - \ell(|x_{mL}^* - x_i|)}{\ell(x_i - x_i) - \ell(|x - x_i|)} \geq \frac{\ell(x_i - x_{mL}) - \ell(x_{mL}^* - x_{mL})}{\ell(x_i - x_{mL}) - \ell(|x - x_{mL}|)}
\]
for both \(x_i \leq x_{mL}, x \geq x_{mL}^*\) and \(x_i \geq x_{mL}, x \leq x_{mL}^*\). This means that if the median primary voter prefers a more moderate candidate to a more extreme one, all primary voters to his right also prefer that more moderate candidate. Similarly, if the median primary voter prefers a more extreme candidate to a more moderate one, all primary voters to his left also prefer that more extreme candidate. It must then be that the median primary voter’s optimal candidate is the Condorcet winner. □

**Proof of Corollary 1.** Replacing \(\ell\) with either absolute or exponential loss in Equation 4 gives
\[
F \left( \frac{x_i + x^*}{2} \right) = \frac{\ell(x_i) - \ell(x^*)}{2\ell'(x^*)}. \tag{5}
\]
This must have a solution with \(x < 0\) by the log-concavity of \(f\), the fact that \(\ell\) is minimized at 0, and a simple application of the intermediate value theorem. The uniqueness follows from the facts that the left-hand side is strictly increasing because of the log-concavity of \(f\), and the
right-hand side is strictly decreasing in $x^*$. Notice that the $x^*$ in Equation (5) does not depend on $x_i$.

Next, recall from Proposition 1 that the optimal candidate of the median primary voter is the Condorcet winner. If $x^*$ is to the right of the median primary voter, then she is $m_L$’s optimal candidate and therefore the Condorcet winner. If $x^*$ is to the left of the median primary voter, $m_L$’s optimal candidate must have the same ideology as him because $\frac{dE_{UL}}{dx} < 0$ for $x \geq x_{m_L}$. Thus, the Condorcet winner must be the either $x_{m_L}$ or $x^*$, whichever is greater. □

**Proof of Proposition 2.** Let $x'_i > x_i$ be the platforms of two incumbents, and $x^{**}$ and $x^*$ challengers chosen by $L$ against $x'_i$ and $x_i$ respectively. We need to show $F\left(\frac{x'_i + x^{**}}{2}\right) \geq F\left(\frac{x'_i + x^*}{2}\right)$. Recall from Lemma 3 that the optimal candidate must be in $[x_i, x_i)$. Notice first that if $x^* = x_i$, it must be that $x^* \leq x^{**}$, and the proposition follows immediately. Thus we only need to prove the proposition for $x^* \in (x_i, x_i)$, which means

$$F\left(\frac{x_i + x^*}{2}\right) 2\ell'(x^* - x_i) = f\left(\frac{x_i + x^*}{2}\right) \left(\ell(x_i - x_i) - \ell(x^* - x_i)\right).$$

must hold with equality. On the other hand, $x^{**}$ could be on a corner or the interior. $x^{**} \in [x_i, x_i)$ requires

$$F\left(\frac{x'_i + x^{**}}{2}\right) 2\ell'(x^{**} - x_i) \geq f\left(\frac{x'_i + x^{**}}{2}\right) \left(\ell(x'_i - x_i) - \ell(x^{**} - x_i)\right).$$

Suppose for a contradiction that $F\left(\frac{x'_i + x^{**}}{2}\right) < F\left(\frac{x_i + x^*}{2}\right)$. By the log-concavity of $f$,

$$\ell'(x^{**} - x_i)(\ell(x_i - x_i) - \ell(x^* - x_i)) > \ell'(x^* - x_i)(\ell(x'_i - x_i) - \ell(x^{**} - x_i)).$$

This requires either $\ell(x^{**} - x_i) > \ell(x^* - x_i)$, or $\ell'(x^{**} - x_i) > \ell'(x^* - x_i)$. Both of these imply $x^{**} > x^*$; former because $\ell$ is increasing, and latter because $\ell$ is convex. But, $x^{**} > x^*$ leads to a contradiction with the premises $x'_i > x_i$ and $F\left(\frac{x'_i + x^{**}}{2}\right) < F\left(\frac{x_i + x^*}{2}\right)$. Thus, it must be that $F\left(\frac{x'_i + x^{**}}{2}\right) \geq F\left(\frac{x_i + x^*}{2}\right)$. □
Proof of Lemma 1. Let us start by restating the sufficient conditions for an incumbent platform \( \tilde{x}_I \) that leaves the median primary voter indifferent to exist. By the differentiability of the loss function and log-concavity of \( f \), we know that the median primary voter’s payoff must be continuous in the incumbent’s platform. Thus, if there is an incumbent against which \( m_L \) prefers \( E \) over \( M \), and another who induces a preference for \( M \) over \( E \), there must then exist \( \tilde{x}_I \) such that he is indifferent between the two. To recover the conditions under which the above premise holds, notice we can write the median primary voter’s net expected utility of \( E \) over \( M \) is \( \Delta_{m_L}(x_I) \).

Recall that for all \( x_I, x_E, \) and \( x_M \), we have \( F\left( \frac{x_I + x_M}{2} \right) \geq F\left( \frac{x_I + x_E}{2} \right) \). Moreover, as \( x_I \to \infty \), by Chebyshev’s Inequality we know that the \( \ell(x_I - x_{m_L}) \left( F\left( \frac{x_I + x_M}{2} \right) - F\left( \frac{x_I + x_E}{2} \right) \right) \) term in \( \Delta_{m_L}(x_I) \) goes to zero, meaning that \( \lim_{x_I \to \infty} \text{EU}_{m_L}(x_E, x_I) - \text{EU}_{m_L}(x_M, x_I) = \ell(x_M - x_{m_L}) - \ell(|x_{m_L} - x_E|) \). If this term is negative, the median primary voter always prefers \( M \) to \( E \), and \( E \) never wins the primary election. The first part of Assumption 1 in the main text ensures that against sufficiently weak incumbents the median primary voter prefers to vote for \( E \).

To rule out the other case where the median primary voter always votes for \( E \), notice that when the incumbent’s platform is equal to zero, the median primary voter’s net expected utility is

\[
\Delta_{m_L}(0) = \left( \ell(-x_{m_L}) - \ell(|x_{m_L} - x_E|) \right) F\left( \frac{x_E}{2} \right) - \left( \ell(-x_{m_L}) - \ell(x_M - x_{m_L}) \right) F\left( \frac{x_M}{2} \right).
\]

It follows that against a very moderate incumbent the median primary voter votes for \( M \) whenever the second part of Assumption 1 holds. Therefore when Assumption 1 holds and so the median primary voter votes for \( M \) against some incumbents and for \( E \) against others, it follows by continuity that there must exist at least one incumbent platform \( \tilde{x}_I \) that leaves him indifferent. Assumption 2 ensures there cannot be multiple such platforms. This is not a critical assumption, and most arguments made below apply to the case when there are multiple incumbent platforms that leave \( m_L \) indifferent between \( E \) and \( M \). Uniqueness of \( \tilde{x}_I \), however, greatly simplifies exposition. Assumption 2 states that the derivative of the expected payoff of the median primary voter with respect to \( x_I \) must be positive when evaluated in the region where he prefers \( E \) to \( M \). In other words, as long as \( m_L \) prefers \( M \), his net payoff from electing \( E \) may increase or decrease as
the incumbent becomes more extreme. But once $m_L$ has a weak preference for $E$, he never goes back to preferring $M$ as the incumbent goes even more extreme.

When Assumptions 1, and 2 hold, there is a unique incumbent platform $\tilde{x}_I$ that leaves the median primary voter indifferent between $E$ and $M$. This platform satisfies $\Delta_{m_L}(\tilde{x}_I) = 0$. Because indifferent voters vote for the more extreme candidate, when the incumbent’s platform is $\tilde{x}_I$, the median primary voter votes for $E$. We know from the proof of Proposition 1 that if the median primary voter votes for $E$ ($M$), then so must all primary voters to his left (right). It follows then when the incumbent’s platform is $x_I < \tilde{x}_I$, the primary winner is $M$; and otherwise it is $E$.

When primary voters choose $E$ as the challenger to face off against the incumbent in the general election, the incumbent’s probability of reelection is $1 - F\left(\frac{x_I + x_E}{2}\right)$. Because $x_E < x_M$, we know that for all $x_I, F\left(\frac{x_I + x_E}{2}\right) < F\left(\frac{x_I + x_M}{2}\right)$. Furthermore, continuity of $f$ ensures the existence of an interval of platforms where the incumbent is reelection with a lower probability than $\tilde{x}_I$ because she faces $M$ instead of $E$. The upper bound of this interval is $\tilde{x}_I$, exclusive, because the probability of reelection is monotonic and continuous in the incumbent’s platform holding fixed the identity of the opponent. To see that the lower bound of this interval is $x_I$, formally define it as $x_I := t^{-1}(\max\{0, \tilde{x}_I + x_E - x_M\})$, and notice that for any platform in the interval $x_I \in (x_I, \tilde{x}_I)$, the challenger is $M$. Because $F\left(\frac{\tilde{x}_I + x_E}{2}\right) < F\left(\frac{x_I + x_M}{2}\right)$ for all $x_I \in (x_I, \tilde{x}_I)$, the lemma obtains. □

**Proof of Proposition 3.** Consider an incumbent whose ideal point is $t \in \left(\frac{x_I + \tilde{x}_I}{2}, \tilde{x}_I\right)$. Taking the derivative of incumbent’s payoff in Equation (3) yields

$$- \frac{1}{2} f\left(\frac{x_I + x_J}{2}\right) \left( B - t(|x_I - t|) \right) - \text{sgn}(x_I - t) t'(|x_I - t|) \left( 1 - F\left(\frac{x_I + x_J}{2}\right) \right),$$

such that $J = E$ if and only if $x_I \geq \tilde{x}_I$. To eliminate potential regions and narrow the set of possible solutions, let us study this derivative separately in the following two regions: $x_I < \tilde{x}_I$ and $x_I \geq \tilde{x}_I$. 

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1. For $x_I \geq \bar{x}_I$, expression (6) becomes negative:

$$-\frac{1}{2} f \left( \frac{x_I + x_E}{2} \right) \left( B - \ell (x_I - t) \right) - \ell' (x_I - t) \left( 1 - F \left( \frac{x_I + x_E}{2} \right) \right).$$

This means that $t = \bar{x}_I$ is strictly preferred to every platform strictly greater than it, and the optimal platform cannot be strictly greater than $\bar{x}_I$.

2. For $x_I < \bar{x}_I$, we can rewrite expression (6) as

$$-\frac{1}{2} f \left( \frac{x_I + x_M}{2} \right) \left( B - \ell (t - x_I) \right) + \ell' (t - x_I) \left( 1 - F \left( \frac{x_I + x_M}{2} \right) \right). \tag{7}$$

First, notice that for $x_I \in (t, \bar{x}_I)$, this expression is always negative. This means that the optimal platform cannot be in this region. Thus, we can restrict focus to $x_I \leq t$.

For sufficiently low $B$, the above expression may be positive for all $x_I \leq t$, which means that the incumbent’s optimal platform in this region is her own ideal point, $t$. The intuition is that when office rents are low and the incumbent is very likely to be reelected with her ideal point, she does not find it worthwhile to moderate her platform to improve her reelection chances. In contrast, for sufficiently high $B$, expression (7) may be negative for all $x_I \leq t$. This implies that the optimal platform for the incumbent is the one that maximizes her probability of reelection at zero. Here, the incumbent always finds it preferable to moderate her platform to improve her reelection chance and obtain large office rents. For intermediate values of $B$, there is an interior optimum that solves

$$B - \ell (t - x_I^{\text{int}}) = 2 \ell' (t - x_I^{\text{int}}) \frac{1 - F \left( x_I^{\text{int}} + x_M \right)}{f \left( \frac{x_I^{\text{int}} + x_M}{2} \right)}. \tag{8}$$

There can be at most one interior solution in this interval. This is because the left-hand side is increasing and the right-hand side is decreasing in $x_I$ as the inverse hazard function on the right inherits log-concavity from $f$ (Bagnoli and Bergstrom, 2005). Let us denote the platform in this region that gives the highest expected utility by $x_I^* \in \{0, x_I^{\text{int}}, t\}$.
Therefore, there are four possible optimal platforms for an incumbent with an ideal point in $(\frac{x_I + \tilde{x}_I}{2}, \tilde{x}_I)$: $x_I = 0$ that maximizes the probability of beating $M$, $x_I = t$ that minimizes policy loss, $x_I = x_I^{\text{int}}$ that satisfies Equation (8), and $x_I = \tilde{x}_I$, the most moderate platform that induces $E$ to win the opposition party primary.

Notice that $x_I^*$ is monotone decreasing in $B$. This is intuitive; as office rents increase the incumbent improves her probability of reelection by moving to the center. Also notice that the cross-partial of the incumbent’s payoff with respect to $x_I$ and $B$ is given by $-\frac{1}{2} f \left( \frac{x_I + x_M}{2} \right)$. It must then be that $x_I^*$ is continuously decreasing in $B$. We can then define $b: (0, t) \to \mathbb{R}_+$ as a surjection that maps incumbent platforms to office rents $B$ that make them optimal for the incumbent, subject to the constraint $x_I \leq t$.

Take $b(x_I)$. By construction, this means that $\text{EU}_I(x_I) \geq \text{EU}_I(x_I)$ for all $x_I \leq t$. But we know by the definition of $x_I$ that $\tilde{x}_I$ results in weakly higher probability of reelection for the incumbent. Furthermore, our restriction of $t > \frac{x_I + x_I}{2}$ ensures that $\tilde{x}_I$ is closer to the incumbent’s ideal point than $x_I$. Therefore, by running on $\tilde{x}_I$ instead, the incumbent can be reelected with as high a probability as $x_I$ and get a policy she strictly prefers. Thus, it follows that for $b(x_I)$, the incumbent’s expected utility is maximized at $x_I = \tilde{x}_I$.

Next, take $B = b(\hat{x}_I)$, where $\hat{x}_I := 2t - \tilde{x}_I$. Again, by construction of $b$ we have $\text{EU}_I(\hat{x}_I) \geq \text{EU}_I(x_I)$ for all $x_I \leq t$. Notice that $\hat{x}_I > x_I$, which implies both that $b(\hat{x}_I) \leq b(x_I)$ because $b$ is decreasing, and that $\hat{x}_I$ leads to a strictly lower probability of reelection for the incumbent than $x_I$. Because $x_I$ leads to the same probability of reelection as $\tilde{x}_I$, it follows that $\hat{x}_I$ leads to a lower probability of reelection than $\tilde{x}_I$ and results in the same policy payoff conditional on election. Thus, when $B = b(\hat{x}_I)$, the incumbent can improve her expected payoff by running on $\tilde{x}_I$ instead. Because we know $\hat{x}_I$ is the constrained optimum, $\tilde{x}_I$ must be the unconstrained optimum.

So we know that for both $b(x_I)$ and $b(\hat{x}_I)$ the incumbent’s optimal strategy is to provoke the opposition by playing $\tilde{x}_I$. The first derivative of the incumbent’s payoff with respect to $B$ is $1 - F \left( \frac{x_I + x_M}{2} \right) \geq 0$. Because the value function is monotone increasing in $B$ in the interval $[0, t]$, it follows by the Envelope Theorem that $\hat{x}_I$ is optimal for all $B \in [b(\hat{x}_I), b(x_I)]$. \qed
Proof of Lemma 2. Under Assumption 3, all arguments from the proof of Lemma 1 involving the existence of an incumbent platform that leaves the extremist indifferent between entering and staying out carry through. When Assumptions 3 and 4 hold, there is a unique platform $\tilde{x}'_I$ which solves $\Delta_E(\tilde{x}'_I) = 0$ and that leaves $E$ indifferent between running and staying out. Candidates run when they are indifferent, meaning that there is a competitive opposition party primary if and only if the incumbent’s platform is at least $\tilde{x}_I$. With probability $p$, the extremist wins a competitive primary and faces the incumbent in the general election. It follows then that an incumbent with platform $x_I \geq \tilde{x}'_I$ faces $E$ with probability $p$ and $M$ with probability $1 - p$. Her probability of being reelected is the sum of the probabilities she faces each candidate times she beats them in the general election, so $p F \left( \frac{\tilde{x}'_I + x_E}{2} \right) + (1 - p) F \left( \frac{\tilde{x}'_I + x_M}{2} \right)$.

Because $F \left( \frac{\tilde{x}'_I + x_M}{2} \right) > F \left( \frac{\tilde{x}'_I + x_E}{2} \right)$ for all $x_I$, it follows by continuity of $\ell$ and $f$ that there exists a some interval to the left of $\tilde{x}'_I$ that lead to a lower probability of incumbent’s reelection.

Next, take $F \left( \frac{x_M}{2} \right)$. If this is less than the probability in expression (9), then $\tilde{x}'_I$ leads to the highest possible reelection probability. If it is larger, then by continuity there must exist a platform $x'_I(p)$ such that

$$F \left( \frac{x'_I(p) + x_M}{2} \right) = p F \left( \frac{\tilde{x}'_I + x_E}{2} \right) + (1 - p) F \left( \frac{\tilde{x}'_I + x_M}{2} \right).$$

It follows that every incumbent with a platform $x_I \in (x'_I(p), \tilde{x}'_I)$ is reelected with a lower probability than $\tilde{x}'_I$.

Proof of Proposition 4. I prove each part of this proposition in the order they are presented. Throughout, I use $\bar{\Delta} := \lim_{x_I \to \infty} \Delta_E(x_I)$, $\bar{\Delta}_M := \sup \{ \Delta_M(x_I) \}$, and $\underline{\Delta}_M := \min \{ \Delta_M(x_I) \}$. Let $2c < \min \{ \underline{\Delta}_M, \bar{\Delta}_E \}$. We know that $\Delta_M, \bar{\Delta}_E \in (0, \infty)$, so this is well-defined. In the first two parts, the order of announcements does not matter because $M$ plays a dominant strategy.

1. To establish that Lemma 2 extends to small positive costs of running, notice that because $\Delta_M$
is bounded away from zero, we can find some \( p \) that satisfies \( p < 1 - \frac{\xi}{\Delta_M} \). This means that for such values of \( p \), \( M \) always runs. Also notice that from Assumption 3, \( \Delta_E(0) < \frac{\xi}{p} \) immediately follows for any \( c, p > 0 \). Finally, because \( f \) has finite variance and \( x_E < x_M \), we can find some \( p > \frac{\xi}{\Delta_E} \). Then, by continuity and Assumption 4 there exists a unique incumbent platform \( \tilde{x}_I' \) that leaves \( E \) indifferent; she enters for \( x_I \geq \tilde{x}_I' \) and stays out otherwise. Thus, for \( p \in \left( \frac{\xi}{\Delta_E}, 1 - \frac{\xi}{\Delta_M} \right) \), we have our result.

2. Let \( p = \frac{\xi}{\Delta_E} \). Then, we have \( \Delta_E(x_I) < \frac{\xi}{p} \) for all \( x_I \geq 0 \) and for any \( p \in [0, \bar{p}) \). This means that \( E \) never enters the race when \( M \) runs. Furthermore, for any \( p \in [0, \bar{p}) \) we have \( \frac{\xi}{1-p} < \Delta_M \leq \Delta_M(x_I) \) for all \( x_I \). It follows that \( M \) always prefers to run, driving \( E \) out. It must then be that \( M \) is the only candidate.

3. Let \( \bar{p} = 1 - \frac{\xi}{\Delta_M} \), and suppose first that \( M \) announces his decision to run, followed by \( E \). Then, for all \( p \in (\bar{p}, 1) \) and \( x_I \), we have \( c < \Delta_M < \Delta_E < \frac{\xi}{1-p} \). This means that \( M \) enters if and only if \( E \) will not join him, and \( E \) can drive \( M \) out of running. Whether she chooses to depends on whether \( \Delta_E(x_I) \) is greater than \( \frac{\xi}{p} \) or not. Notice that we have \( \Delta_E(x_I) > \frac{\xi}{p} \) for \( p \in (\bar{p}, 1) \) sufficiently high \( x_I \) because \( 1 - \frac{\xi}{\Delta_M} > 1 - \frac{\xi}{\Delta_E} > \frac{\xi}{\Delta_E} \). Furthermore, from Assumption 3 it follows that \( \Delta_E(0) < 0 < \frac{\xi}{p} \). By continuity it must be then for some intermediate values of \( \tilde{x}_I' \) such that \( \Delta_E(\tilde{x}_I') = \frac{\xi}{p} \). When the incumbent’s platform is \( \tilde{x}_I' \), \( E \) is indifferent between entering and staying out, and enters. It follows that for \( p \in (\bar{p}, 1) \), against an incumbent with a platform \( x_I < \tilde{x}_I' \) only \( M \) runs, and against an incumbent with a platform \( x_I \geq \tilde{x}_I' \) only \( E \) runs.

Suppose now \( E \) announces first, and \( M \) second. Here, \( E \) runs if and only if she prefers facing the incumbent herself rather than \( M \). As before, for all \( p \in (\bar{p}, 1) \) we have \( c < \Delta_M < \Delta_E < \frac{\xi}{1-p} \), and so \( M \) enters if and only if \( E \) has stayed out. \( E \) enters when \( \Delta_E(x_I) \geq c \). For sufficiently high values of \( x_I \) this must hold because \( 2c < \Delta_E \). Again, from Assumption 3 it follows that \( \Delta_E(0) < 0 < c \). Then, by Assumption 4 there exists a unique incumbent platform \( \tilde{x}_I' \) such that when \( p \in (\bar{p}, 1) \), against an incumbent with a platform \( x_I < \tilde{x}_I' \) only \( M \) runs and against an incumbent with a platform \( x_I \geq \tilde{x}_I' \) only \( E \) runs.
Proof of Proposition 5. When there is a primary advantage for extremists, we know from Lemma 2 that depending on the order of announcements, there exists a unique incumbent platform $\tilde{x}_I'$ such that for $x_I \geq \tilde{x}_I'$ the challenger is $E$, and for $x_I < \tilde{x}_I'$ the challenger is $M$. Define as before $x_I := t^{-1}(\max\{0, \tilde{x}_I' + x_E - x_M\})$, and take $t \in \left(\frac{\tilde{x}_I' + x_E}{2}, \tilde{x}_I'\right)$. The proof of Proposition 3 carries through with $\tilde{x}_I'$ replacing $\tilde{x}_I$.

Suppose now there is no primary advantage for either side. Then, we know from Lemma 2 that $M$ always runs regardless of $x_I$, and that there exists a unique incumbent platform $\tilde{x}_I'$ such that $E$ enters the race alongside $M$ if and only if $x_I \geq \tilde{x}_I'$. When $E$ enters, she wins the primary and becomes the challenger with probability $p$. Thus, the probability of reelection for the incumbent is $F\left(\frac{x_I + x_M}{2}\right)$ for $x_I < \tilde{x}_I'$, and $pF\left(\frac{x_I + x_E}{2}\right) + (1-p)F\left(\frac{x_I + x_M}{2}\right)$ for $x_I \geq \tilde{x}_I'$. Take an incumbent with ideal point $t \in \left(\frac{x_I(p) + \tilde{x}_I}{2}, \tilde{x}_I\right)$, where $x_I(p)$ is defined as in the text. Once again, the proof of Proposition 3 carries through with $\tilde{x}_I'$ replacing $\tilde{x}_I$, and $x_I(p)$ replacing $x_I$.

That provoking the opposition cannot occur when there is a primary advantage for moderates follows from the fact that $E$ never runs, and $M$ always runs when $p$ is sufficiently low. □

Appendix B  Combining the two Models

Here, I present results from simulations of a model that has both primary voters, and endogenous entry decisions by candidates. To ensure probabilities of winning the primary are on the interior for some parameter values, I assume here that there is also uncertainty about the ideal point of the median primary voter. Specifically, I assume that the ideal point of the median primary voter is drawn from some log-concave distribution bounded above by zero and finite variance.

Simulations presented on Figure 6 show that against very moderate incumbents, primary voters support the moderate candidate. When the incumbent is more extreme, the probabilities $M$ and $E$ would beat her start converging. This leads the primary voters with ideal points sufficiently to the left to start supporting the extremist. Finally, when the incumbent is very extreme, the
Figure 6: Orange (SW) and blue (NE) regions respectively correspond to parameter values where only \( M \) and \( E \) run in equilibrium. In green (E) both candidates run. The median voter’s ideal point is standard normally distributed. Losses are linear on the left plot and quadratic on the right. Parameter values are \( x_E = -1.5, x_M = -0.5, c = 1.2, \) and \( B = 4 \). Lines refer to the probabilities \( E \) wins the primary when the ideal point of the median primary voter is drawn from a normal distribution with a standard deviation equal to 1, and means \(-1.3, -1.1, \) or \(-0.8, \) depicted by the orange, blue, and red lines respectively on the left plot, and means \(-2.5, -1.2, \) and \(-0.1 \) on the right.

The net payoff for \( E \) over \( M \) of primary voters with ideal points close to \( E \) is very similar to \( \Delta_E(x_I) \). This is intuitive, the extreme opposition primary voters and candidates face similar problems and have similar payoffs, with the exception that the candidate also cares about office rents and costs of running for office. As such, extremist primary voters support the extremist candidate in similar conditions as when she wants to enter the race. It follows that if the median primary voter has an ideal point close to \( x_E \), incumbent moving away from the center increases both the probability \( E \) wins in the general election, and the probability she wins in the primary election. Thus, the effect of the incumbent’s platform on the primary voters’ calculus reinforces the extremist candidate’s entry decision.
Appendix C  Supplementary Figures

Here, I reproduce Figure 4 for when $E$ moves first instead to show that the order of announcements does not lead to significant changes in who runs in equilibrium. Notice that unless $p$ is close to one, running is a dominant strategy for $M$, and thus he runs regardless of the sequence of announcements. The order only matters when both candidates prefer to be the challenger themselves, but not so much to induce a competitive primary where they might lose. This can only happen when $p$ is high and so $(1 - p)\Delta_M(x_I) < c$, meaning that $M$ wants to stay out when $E$ enters. The condition for $E$ to be only candidate running in equilibrium when $M$ moves first is $p\Delta_E(x_I) > c$, that is, $E$ prefers to run even when $M$ is running, and so she only wins the primary with probability $p$. The same condition when $E$ moves first is $\Delta_E(x_I) > c$, because she knows her entry will deter $M$. Thus, the only case where the identity of the challenger depends on the order is $p\Delta_E(x_I) < c < \Delta_E(x_I)$ and $(1 - p)\Delta_M(x_I) < c$. The latter condition requires $p$ to be close to one, which means the region where the identity of the challenger is sensitive to the order of announcements must be narrow.

Figure 7: Orange (SW) and blue (NE) regions respectively correspond to parameter values where only $M$ and $E$ run in equilibrium. In green (E) both candidates run. The median voter’s ideal point is standard normally distributed. Losses are linear on the left plot and quadratic on the right. Parameter values are $x_E = -1.5$, $x_M = -0.5$, $c = 1.2$, and $B = 4$. $E$ moves first.